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A p-adic variant of Kontsevich–Zagier integral operation rules and of Hrushovski–Kazhdan style motivic integration

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Dedicated to Angus Macintyre, source of inspiration

Abstract. We prove that if two semi-algebraic subsets of $\mathbb{Q}_p^n$ have the same p-adic measure, then this equality can already be deduced using only some basic integral transformation rules. On the one hand, this can be considered as a positive answer to a p-adic analogue of a question asked by Kontsevich–Zagier in the reals (though the question in the reals is much harder). On the other hand, our result can also be considered as stating that over $\mathbb{Q}_p$, universal motivic integration (in the sense of Hrushovski–Kazhdan) coincides with the usual p-adic integration.

1. Introduction

A period is a real number that can be obtained by integrating a rational function f over a semi-algebraic domain $X \subseteq \mathbb{R}^n$, where both f and X are defined with coefficients in $\mathbb{Q}$. Kontsevich and Zagier [13] put forward the question whether manipulating real integrals by basic rules like Stokes, semi-algebraic change of variables, and linearity can explain all equalities between periods, possibly even in an algorithmic, decidable way. On a different matter, motivic integration is an abstract notion of integration in certain settings where no reasonable Lebesgue measure exists, but which satisfies similar basic rules as classical integration. One crucial insight which led to the version of motivic integration by Hrushovski and Kazhdan [12]

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(in algebraically closed valued fields of equi-characteristic 0) is that the motivic integral can simply be defined as the universal map satisfying those basic rules. Here, universality means that no more relations hold than necessary: An equality between motivic integrals holds if and only if it is implied by the basic manipulation rules. In other words, the Kontsevich–Zagier Conjecture is true for this version of motivic integration by definition.

The main result of the present paper is a positive answer to a p-adic analogue of the Kontsevich–Zagier question, where one integrates over semi-algebraic domains in $\mathbb{Q}_p^n$ with respect to the p-adic measure: Any equality between such p-adic integrals can be deduced using a few specific basic manipulations. Note however that whereas the original question by Kontsevich and Zagier has deep links to transcendental number theory, those links are completely lost in our p-adic case; notably, the p-adic measure of any p-adic semi-algebraic set is always a rational number.

In universal (Hrushovski–Kazhdan style) motivic integration, a central difficulty consists in describing the target ring of integration (more) explicitly. Hrushovski and Kazhdan do this in the case of algebraically closed valued fields of equi-characteristic 0 (and then generalize this to Henselian valued fields of equi-characteristic 0), but the case of valued fields of mixed characteristic remains open. From that point of view, our result that Kontsevich–Zagier holds in $\mathbb{Q}_p$ just means that the target ring of universal motivic integration over $\mathbb{Q}_p$ is simply $\mathbb{Q}$, and that motivic integration is the same as Lebesgue integration with respect to the Haar measure.

Let us explain our main result in more detail. Suppose that $X_1 \subseteq \mathbb{Q}_p^{n_1}$, $X_2 \subseteq \mathbb{Q}_p^{n_2}$ are definable sets in the language of valued fields (also called semi-algebraic sets in the context of valued fields) such that $\mu(X_i)$ is finite for $i = 1, 2$, where μ denotes the Haar measure on $\mathbb{Q}_p^{n_i}$, normalized so that $\mathbb{Z}_p^{n_i}$ has measure 1. Does $\mu(X_1) = \mu(X_2)$ imply that X_1 can be transformed into X_2 by some basic rules? To make the question precise, we define a ring $R_{\{0\}}$ generated by sets X_i as above, and quotient by relations corresponding to natural integral transformations (see Definition 2.1). A positive answer to the question then corresponds to: The map from this ring $R_{\{0\}}$ to $\mathbb{R}$ sending the class of $X \subseteq \mathbb{Q}_p^n$ to its measure $\mu(X)$ is injective; this is the statement of Corollary 2.4. More precisely, as mentioned above, it is known that definable sets have rational measure, so that we obtain an isomorphism $R_{\{0\}} \cong \mathbb{Q}$.

One main result of Hrushovski and Kazhdan is a description of the semi-ring denoted by $K_+(\mu_\Gamma \mathrm{VF})[*]$ (see [12, Theorem 1.2]). Our ring $R_{\{0\}}$ is closely related to this and plays a similar role. Apart from $R_{\{0\}}$ being the *ring* generated by *bounded* definable sets (whereas $K_+(\mu_\Gamma \mathrm{VF})[*]$ is a *semi-ring* generated by *arbitrary* definable sets), the differences are mainly of technical nature. In particular, we can avoid some technicalities using that we only work in $\mathbb{Q}_p$. We further simplified the definition of $R_{\{0\}}$ by not using grading by dimension.

The original question of Kontsevich and Zagier was not just about measures of sets X, but about integrals of functions on X. However, it is not very difficult to get from one version of the question to the other, using that integrals can be expressed in terms of measures of sets; see the explanations below Corollary 2.4 for some details.

Our main result – Theorem 2.3 – is a family version of Corollary 2.4, namely: Given two definable families $(X_{i,s})_{s\in S}$ of sets, if $\mu(X_{1,s}) = \mu(X_{2,s})$ for each $s \in S$, then all transformations needed to turn $X_{1,s}$ into $X_{2,s}$ can be carried out uniformly in s (i.e., with definable families of transformations). This is made precise by introducing a ring R_S which is a family variant of the above ring $R_{\{0\}}$. The proof builds on a similar kind of result in the value group, obtained in [6], see Theorem 3.16 below.

One should also compare the basic transformation rules (R1)–(R4) and Corollary 2.4 with the classification of definable sets without measure from [1], where it is shown that two infinite definable subsets of $\mathbb{Q}_p^n$ are definably isomorphic (namely, in definable bijection) if and only if they have the same dimension. The question to classify integrals rather than definable sets without measure was raised by Angus Macintyre in 2001, after [1]. We thank him for raising this question.

We of course rely on classical model theory of $\mathcal{L}$-definable sets enabled by the quantifier elimination result by Macintyre [14], cell decomposition by Denef [10], and dimension theory by van den Dries [17], to reduce to Presburger sets [15]. To go beyond the situation treated in this paper, we mention the upcoming analogue of the rings R_S in elementary extensions of $\mathbb{Q}_p$ in work in progress in the PhD thesis of Florian Severin. (Note that there is no Haar measure on non-standard elementary extensions.) Other generalizations, e.g., to the framework of motivic integrals from [8] and finding adequate integral operation rules, are left for the future. A more direct generalization to any finite field extension of $\mathbb{Q}_p$ and to other languages than $\mathcal{L}$ is formulated in the final Remark 4.5.

2. Precise statement of the main results

Let $\mathcal{L} = \{+, \cdot, \mathcal{O}\}$ be the language of valued fields, namely, with the ring operations and a predicate for the valuation ring. By an $\mathcal{L}$-definable set we mean a subset $X \subseteq \mathbb{Q}_p^n$ for some n which is given by a parameter free $\mathcal{L}$-formula φ. A function between $\mathcal{L}$-definable sets is called $\mathcal{L}$-definable if its graph is an $\mathcal{L}$-definable set. (Remark 4.5 specifies some variants of the language for which our results also hold.)

When $X \subseteq \mathbb{Q}_p^n$ is a Borel-measurable set, then we write $\mu(X) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ for the measure of X with respect to the Haar measure on (the additive group) $\mathbb{Q}_p^n$, normalized in such a way that $\mu(\mathbb{Z}_p^n) = 1$. (This naturally also makes sense for $n = 0$, where $\mathbb{Q}_p^0 = \mathbb{Z}_p^0$ is a one-point set, namely containing the empty tuple.) It follows, e.g., from Macintyre's quantifier elimination result [14] that any $\mathcal{L}$-definable set is measurable.

Definition 2.1. Fix an $\mathcal{L}$-definable set S. We let R_S be the abelian group generated by $\mathcal{L}$-definable sets $X \subseteq S \times \mathbb{Q}_p^n$ (for all $n \geq 0$) such that $X_s := \{x \in \mathbb{Q}_p^n \mid (s, x) \in X\}$ has finite measure for each $s \in S$, modulo the following relations, and where we write $[X]$ for the element of R_S corresponding to the $\mathcal{L}$-definable set X.

(R1) (Additivity) If X_1 and X_2 are disjoint $\mathcal{L}$-definable subsets of $S \times \mathbb{Q}_p^n$, then

$$[X_1 \cup X_2] = [X_1] + [X_2].$$

(R2) (Negligible sets) If $X_s \subseteq \mathbb{Q}_p^n$ has dimension strictly less than n for each s in S, then

$$[X] = 0.$$

(R3) (Change of variables) Suppose that $\phi \colon X \to Y$ is an $\mathcal{L}$-definable bijection between $\mathcal{L}$-definable sets $X, Y \subseteq S \times \mathbb{Q}_p^n$ inducing a bijection $\phi_s \colon X_s \to Y_s$ for each $s \in S$. Suppose moreover that the sets X_s and Y_s are open in $\mathbb{Q}_p^n$, that ϕ_s is C^1, and that the (p-adic) norm of the Jacobian determinant of ϕ_s equals 1 everywhere, for each s in S. Then

$$[X] = [Y].$$

(R4) (Product with unit ball) For any $\mathcal{L}$-definable $X \subseteq S \times \mathbb{Q}_p^n$,

$$[X] = [X \times \mathbb{Z}_p].$$

The group R_S can be endowed with the structure of commutative ring with unit, by defining $[X] \cdot [Y]$ as $[X \times_S Y]$, the class of the fiber product over S; see Lemma 3.4.

For each fixed $s \in S$, the map sending an $\mathcal{L}$-definable set $X \subseteq S \times \mathbb{Q}_p^n$ to the p-adic measure $\mu(X_s)$ factors over relations (R1)–(R4) (note that in (R3), $\mu(X_s) = \mu(Y_s)$ follows from the p-adic version of the change of variables formula for integrals). Therefore, this induces a map from R_S to $\mathbb{R}$, which clearly is a ring homomorphism:

Definition 2.2. Given a definable set S and an element $s \in S$, we denote by $\mu_s \colon R_S \to \mathbb{R}$ the (unique) ring homomorphism sending the class $[X]$ of an $\mathcal{L}$-definable set $X \subseteq S \times \mathbb{Q}_p^n$ to the p-adic measure $\mu(X_s)$ of its fiber at s.

Now we can formulate our main result.

Theorem 2.3. *Fix an $\mathcal{L}$-definable subset S of some cartesian power of $\mathbb{Q}_p$. Then the ring homomorphism which sends an element $\Xi \in R_S$ to the function $S \to \mathbb{R}, s \mapsto \mu_s(\Xi)$ is injective.*

It is well known (and can be deduced from cell decomposition [10]) that the p-adic measure of any definable set is a rational number. In particular, the ring homomorphism from the above theorem takes values in the functions from S to $\mathbb{Q}$. Describing its image precisely would be possible but rather technical. However, in the special case that S is a singleton (e.g., $S = \{0\}$), we can easily be more precise:

Corollary 2.4. *The map sending $[X]$ in $R_{\{0\}}$ to $\mu(X)$ is an isomorphism $R_{\{0\}} \to \mathbb{Q}$ of rings.*

(The deduction of this corollary from the theorem is given after Remark 3.12.)

As mentioned in the introduction, one can formulate a variant of the above results, where measures of sets are replaced by integrals of functions. One way to do this consists in considering a variant R'_S of the above ring R_S which is generated by pairs (X, f), for $X \subseteq S \times \mathbb{Q}_p^n$ and $f \colon X \to \mathbb{Q}_p$ $\mathcal{L}$-definable for which $\int_{x \in X_s} |f(s,x)|$ is finite for each s, where $|y| = p^{-v(y)}$ is the p-adic norm with $v : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ the p-adic valuation and where the integral is with the Haar measure μ on $\mathbb{Q}_p^n$. The relations of R'_S are natural analogues of the ones of R_S, where in (R3), we replace the assumption that the Jacobian determinant of ϕ_s has norm 1 by the relation between the function f on X and the function g on Y coming from the change of variables formula for the transformations $\phi_s : X_s \to Y_s$, namely, the relation

$$|g \circ \phi(s,x)| \cdot |\mathrm{Jac}(\phi_s)(x)| = |f(s,x)|$$

for all x in X_s, with $\mathrm{Jac}(\phi_s)$ the Jacobian determinant of ϕ_s.

One can easily deduce the analogue of Theorem 2.3 for R'_S from the original form of the theorem for R_S. To this end, one notes that the natural ring homomorphism ρ_S from R_S to R'_S sending $[X]$ to $[X, 1]$ is an isomorphism, with 1 being the constant function 1 on X.

Indeed, using the R'_S-version of (R3) and (R4), one easily obtains that $[X, f] = [Y, 1]$ in R'_S, with $Y := \{(x, y) \in X \times \mathbb{Q}_p \mid |y| \leq |f(x)|\}$. The surjectivity of ρ_S follows. The injectivity of ρ_S follows from Theorem 2.3 for R_S.

The proof of Theorem 2.3 is carried out in Section 4; the strategy is as follows. Firstly, one uses Cell Decomposition to reduce to the case of certain definable sets $P(\Lambda)$ that are entirely described in terms of a Presburger definable set $\Lambda \subseteq \mathbb{Z}^n$. (Some variants of those $P(\Lambda)$ are also used.) Then one further reduces to particularly simple such sets (called "basic sets" in Definition 4.1). Those two reduction steps are carried out in the proof of Proposition 4.4.

The case of basic sets is treated in Proposition 4.3: One reduces the problem of understanding equalities in R_S to questions involving only Presburger definable sets, and those questions have already been answered in [6].

This proof strategy requires a good understanding of the classes in R_S of the sets $P(\Lambda)$ (and of their variants). This understanding is developed in Section 3.

3. Sets defined in terms of Presburger data

From now on, S will almost always be a fixed $\mathcal{L}$-definable subset of a cartesian power of $\mathbb{Q}_p$, and s will always be a variable running over S. We also use the following conventions:

Convention–Remark 3.1. We will often identify an $\mathcal{L}$-definable set $X \subseteq S \times \mathbb{Q}_p^n$ with the family $(X_s)_{s \in S}$, and we call a family of subsets of $\mathbb{Q}_p^n$ arising in this way an $\mathcal{L}$-definable family (parametrized by S). Similarly, a family of maps $(f_s)_{s \in S}$ with $f_s \colon X_s \to Y_s$ is called an $\mathcal{L}$-definable family if it arises from an $\mathcal{L}$-definable map $f \colon X \to Y$, with $X \subseteq S \times \mathbb{Q}_p^n$, $Y \subseteq S \times \mathbb{Q}_p^m$.

We consider the value group $\mathbb{Z}$ (together with ∞) as an imaginary sort, i.e., we call a subset of $\mathbb{Q}_p^n \times (\mathbb{Z} \cup \{\infty\})^m$ an imaginary $\mathcal{L}$-definable set if its preimage in $\mathbb{Q}_p^{n+m}$ (under the map sending the last m coordinates to their valuation) is $\mathcal{L}$-definable. We may sometimes drop the word imaginary if it is clear from the context. (The notions of families are also applied in this generalized setting.)

By Quantifier Elimination [14], $X \subseteq \mathbb{Z}^n$ is an imaginary $\mathcal{L}$-definable set if and only if it is a Presburger set, i.e., definable in the language $(+, <)$ of ordered abelian groups; we will sometimes also use this terminology.

Notation 3.2. For each integer $\ell > 0$, we write

$$\mathrm{ac}_\ell : \mathbb{Q}_p \to \mathbb{Z}_p / p^\ell \mathbb{Z}_p$$

for the map sending 0 to 0 and non-zero x to $x p^{-v(x)} \bmod p^\ell \mathbb{Z}_p$. We also write ac for ac_1. Given $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$, we write

$$v(x) := (v(x_1), \ldots, v(x_n)) \in (\mathbb{Z} \cup \{\infty\})^n$$

for the coordinatewise p-adic valuation map.

Relation (R3) in Definition 2.1 expresses that the class in R_S of a definable set is preserved by a measure-preserving bijection. To formally work with relation (R3), we introduce the terminology "(R3)-measure-preserving" for maps ϕ satisfying the conditions of (R3); more precisely:

Definition 3.3. We say that an $\mathcal{L}$-definable map $\phi\colon X \to Y$ between $\mathcal{L}$-definable sets $X, Y \subseteq S \times \mathbb{Q}_p^n$ is (R3)-measure-preserving if for each $s \in S$, it induces a bijection $\phi_s\colon X_s \to Y_s$ which is C^1 and such that the (p-adic) norm of the Jacobian determinant of ϕ_s is equal to 1 everywhere. (For C^1 to make sense, we assume that X_s is open.)

Let us first state and prove that the group R_S from Definition 2.1 carries a natural ring structure.

Lemma 3.4. *Let S be an $\mathcal{L}$-definable set. The group R_S (from Definition* 2.1*) becomes a commutative ring with unit element $[S \times \mathbb{Z}_p]$ and with multiplication induced by*

$$[X] \cdot [Y] := [X \times_S Y],$$

where $X \times_S Y$ is the fiber product over S, namely $\{(s, x, y) \mid s \in S,\ x \in X_s,\ y \in Y_s\}$.

Proof. Let $\tilde{R}_S$ be the quotient of the free abelian group generated by same definable sets as in Definition 2.1, modulo the relations $[X] = [Y]$ whenever $X, Y \in S \times \mathbb{Q}_p^n$ are definable sets such that Y is obtained from X by a permutation of the last n coordinates. It is clear that this group becomes a commutative ring using the above multiplication, and it is then equally easy to verify that the relations from Definition 2.1 form an ideal in $\tilde{R}_S$, which finishes the proof. □

Remark 3.5. Since R_S is a ring with unit, we have a unique ring homomorphism $\mathbb{Z} \to R_S$ (preserving the unit). This ring homomorphism is injective; indeed, consider for any (fixed) s in S the ring homomorphism $\mu_s : R_S \to \mathbb{R}$ and restrict it to the image of $\mathbb{Z}$ inside R_S. We will often identify $\mathbb{Z}$ with its image in R_S.

Remark 3.6. Any finite sum $\sum_i [X_i] \in R_S$ is equal to an element of the form $[Y]$. Indeed, using (R4), we can first assume that $X_i \subseteq S \times \mathbb{Q}_p^n \times \mathbb{Z}_p$ for all i (where n does not depend on i). Then we can translate the last coordinate of each X_i to make them all pairwise disjoint (using (R3)), and then we apply (R1). In particular, every element of R_S can be written in the form $[X] - [Y]$. Also note that one does not need to be careful concerning the order of coordinates (e.g., in (R4)), since (R3) includes coordinate permutations.

Remark 3.7. For disjoint $\mathcal{L}$-definable sets S, S', we have a natural ring isomorphism $R_{S \cup S'} \cong R_S \times R_{S'}$.

Definition 3.8. Given an imaginary $\mathcal{L}$-definable set $\Lambda \subseteq S \times \mathbb{Z}^n$ (for some $n \geq 0$), we define a set $P(\Lambda) \subseteq S \times \mathbb{Q}_p^n$ via its fibers over S:

$$P(\Lambda)_s = \{x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n \mid \mathrm{ac}(x_1) = \cdots = \mathrm{ac}(x_n) = 1,\ v(x) \in \Lambda_s\}, \quad s \in S.$$

If, in addition, we are given an imaginary $\mathcal{L}$-definable function ν to $\mathbb{Z}$ whose domain contains Λ, we similarly define $P(\Lambda, \nu) \subseteq S \times \mathbb{Q}_p^n$ via

$$P(\Lambda, \nu)_s = \left\{ (x, y) \in P(\Lambda)_s \times \mathbb{Q}_p \;\middle|\; \mathrm{ac}(y) = 1,\ v(y) = -n - 1 - \nu_s(v(x)) - \sum_{i=1}^n v(x_i) \right\}.$$

If $n = 1$ and an integer $\ell > 0$ is given, then define furthermore $P_\ell(\Lambda)$ via

$$P_\ell(\Lambda)_s = \{x \in P(\Lambda)_s \mid \mathrm{ac}_\ell(x) = 1\}$$

and $P_\ell(\Lambda, \nu)$ via

$$P_\ell(\Lambda, \nu)_s := P(\Lambda, \nu)_s \cap P_\ell(\Lambda)_s \times \mathbb{Q}_p.$$

Clearly, the sets $P(\Lambda)$, $P(\Lambda, \nu)$, $P_\ell(\Lambda)$ and $P_\ell(\Lambda, \nu)$ are definable. The motivation behind this definition of $P(\Lambda, \nu)$ is that it is a simple way of defining a set with a prescribed p-adic measure, namely (as an easy computation shows)

$$\mu(P(\Lambda, \nu)_s) = \sum_{\lambda \in \Lambda_s} p^{\nu_s(\lambda)}. \tag{3.1}$$

We will see, in several of the following lemmas, that the classes in R_S of sets of the form $P(\Lambda, \nu)$ satisfy relations one would expect from (3.1) (provided that $\mu(P(\Lambda, \nu)_s)$ is finite for all s).

Lemma 3.9. *Let Δ_n be the image of the diagonal embedding of the non-negative integers $\mathbb{N}$ into $\mathbb{Z}^n$, i.e.,*

$$\Delta_n = \{(\underbrace{\lambda, \dots, \lambda}_{n}) \in \mathbb{Z}^n \mid \lambda \in \mathbb{N}\}.$$

Then we have $(p^n - 1) \cdot [P(S \times \Delta_n)] = 1$ in the ring R_S.

Proof. For any $a \in \mathbb{F}_p^n \setminus \{0\}$, there exists a matrix $M \in \mathrm{GL}_n(\mathbb{Z}_p)$ such that $\mathrm{res}(M)$ sends the vector $(1, \dots, 1) \in \mathbb{F}_p^n$ to a. Such an M is (R3)-measure-preserving, and it sends $P(\Delta_n)$ to

$$X_a := \{(p^\lambda x_1, \dots, p^\lambda x_n) \mid \lambda \in \mathbb{N},\ x_i \in \mathrm{res}^{-1}(a_i)\}.$$

By (R3), we obtain $[S \times X_a] = [S \times P(\Delta_n)]$, and since the ($p^n - 1$ many) sets X_a (for a as above) form a partition of $\mathbb{Z}_p^n \setminus \{0\}$, we obtain (in R_S):

$$(p^n - 1)[P(S \times \Delta_n)] \overset{\text{(R1)}}{=} [S \times (\mathbb{Z}_p^n \setminus \{0\})] \overset{\text{(R2)}}{=} [S \times \mathbb{Z}_p^n] \overset{\text{(R4)}}{=} 1. \qquad \square$$

Lemma 3.10. *Consider an imaginary $\mathcal{L}$-definable subset $\Lambda \subseteq S \times \mathbb{Z}$, an integer $\ell > 0$ and an imaginary $\mathcal{L}$-definable function ν to $\mathbb{Z}$ whose domain contains Λ such that $P_\ell(\Lambda)_s$ and $P_\ell(\Lambda, \nu)_s$ have finite measure for each s in S. Then*

$$p^{\ell-1}[P_\ell(\Lambda)] = [P(\Lambda)] \quad \textit{and} \quad p^{\ell-1}[P_\ell(\Lambda, \nu)] = [P(\Lambda, \nu)]$$

holds in R_S.

Proof. We choose representatives $r_1, \dots, r_{p^{\ell-1}}$ of the different cosets of the quotient $(1 + p\mathbb{Z}_p)/(1 + p^\ell\mathbb{Z}_p)$. Then multiplication by r_i is (R3)-measure-preserving and for each s, the sets $r_i P_\ell(\Lambda)_s$ form a partition of $P(\Lambda)_s$. Thus the claim follows from (R3) and (R1).

The same proof also gives the second part, if in "$r_i P_\ell(\Lambda, \nu)_s$", one lets r_i act on the first coordinate only. $\square$

Lemma 3.11. *There exists a (unique) injective ring homomorphism $\mathbb{Q} \to R_S$. In particular, the additive group of R_S is divisible and torsion free.*

Proof. By Remark 3.5, we have $\mathbb{Z} \subseteq R_S$, so it suffices to prove that R_S contains a multiplicative inverse of ℓ for every prime ℓ. (Indeed, torsion freeness then follows by multiplying both sides of an equation of the form "$n \cdot \Xi = 0$" (where $n \geq 1$ and $\Xi \in R_S$) by $\frac{1}{n}$.)

Since $\mathbb{Z}_p$ is the disjoint union of p translates of $p\mathbb{Z}_p$, we have $p \cdot [S \times p\mathbb{Z}_p] = 1$, so that $[S \times p\mathbb{Z}_p]$ is a multiplicative inverse of p. If $\ell \neq p$, then Lemma 3.9 provides the desired multiplicative inverse (namely, a multiple of $[P(S \times \Delta_n)]$), provided that we can find an $n \geq 1$ such that ℓ divides $p^n - 1$. Indeed, the image of p in the ring $\mathbb{Z}/\ell\mathbb{Z}$ is a unit, so for n the order of that image in the group $(\mathbb{Z}/\ell\mathbb{Z})^\times$ of units, we obtain $p^n \equiv 1 \bmod \ell$ and hence ℓ divides $p^n - 1$. □

We will from now on identify $\mathbb{Q}$ with its image in R_S.

Remark 3.12. We now easily see that the class in R_S of a ball in $\mathbb{Q}_p$ is the expected one, namely: The above proof in particular yields that $[S \times p\mathbb{Z}_p] = \frac{1}{p}$. In a similar way, we deduce $[S \times p^c\mathbb{Z}_p] = p^{-c}$ for any integer c, and then also $[S \times (a + p^c\mathbb{Z}_p)] = p^{-c}$ for any $a \in \mathbb{Q}_p$ (by applying (R3) to the translation by a).

We now have all the ingredients to deduce Corollary 2.4 from Theorem 2.3:

Proof of Corollary 2.4 *using Theorem* 2.3. By Theorem 2.3, the map from $R_{\{0\}}$ to $\mathbb{Q}$ induced by the Haar measure is injective. Its restriction to $\mathbb{Q} \subseteq R_{\{0\}}$ is clearly the identity, so surjectivity follows. □

Lemma 3.13. *For $i = 1, 2$ and $n_i \in \mathbb{N}$, let $\Lambda_i \subseteq S \times \mathbb{Z}^{n_i}$ be imaginary $\mathcal{L}$-definable sets, let $\nu_i \colon \Lambda_i \to \mathbb{Z}$ be imaginary $\mathcal{L}$-definable functions, and define $\nu \colon \Lambda_1 \times_S \Lambda_2 \to \mathbb{Z}$ by $(s, \lambda_1, \lambda_2) \mapsto \nu_1(s, \lambda_1) + \nu_2(s, \lambda_2)$. Suppose that $P(\Lambda_i, \nu_i)_s$ has finite measure for $i = 1, 2$ and each s in S. Then so do the $P(\Lambda_1 \times_S \Lambda_2, \nu)_s$ for all s and we have*

$$[P(\Lambda_1 \times_S \Lambda_2, \nu)] = [P(\Lambda_1, \nu_1)] \cdot [P(\Lambda_2, \nu_2)]$$

in R_S.

Remark 3.14. Some useful special cases are obtained for $n_2 = 0$: Let $\Lambda \subseteq S \times \mathbb{Z}^n$ and $\nu \colon \Lambda \to \mathbb{Z}$ be $\mathcal{L}$-definable, with $\mu(P(\Lambda, \nu)_s)$ finite for every $s \in S$.

(1) If ν is of the form $\nu(s, \lambda) = \nu'(s)$ for some $\nu' \colon S \to \mathbb{Z}$, then

$$[P(\Lambda, \nu)] = [P(\Lambda \times_S S, \nu)] = [P(\Lambda, 0)] \cdot [P(S, \nu')].$$

(2) If c is an integer, then

$$[P(\Lambda, \nu + c)] = [P(\Lambda \times_S S, \nu + c)] = [P(\Lambda, \nu)] \cdot [P(S, c)] = p^c \cdot [P(\Lambda, \nu)],$$

where the last equality holds by Remark 3.12.

Proof of Lemma 3.13. To prove the lemma, we will specify an $\mathcal{L}$-definable family of (R3)-measure-preserving bijections

$$P(\Lambda_1 \times_S \Lambda_2, \nu)_s \times \mathbb{Z}_p \to P(\Lambda_1, \nu_1)_s \times P(\Lambda_2, \nu_2)_s.$$

The lemma then follows from (R3), (R4), and the definition of the product.

Fix $s \in S$. We have two natural projections:

$$\pi_R\colon P(\Lambda_1 \times_S \Lambda_2, \nu)_s \to P(\Lambda_1 \times_S \Lambda_2)_s$$

and

$$\pi_L\colon P(\Lambda_1, \nu_1)_s \times P(\Lambda_2, \nu_2)_s \to P(\Lambda_1)_s \times P(\Lambda_2)_s = P(\Lambda_1 \times_S \Lambda_2)_s.$$

For $x \in P(\Lambda_1 \times_S \Lambda_2)_s$, the fibers over x are of the form

$$\pi_R^{-1}(x) = p^{\sigma(s,x)} + p^{\sigma(s,x)+1}\mathbb{Z}_p$$

and

$$\pi_L^{-1}(x) = (p^{\sigma_1(s,x)} + p^{\sigma_1(s,x)+1}\mathbb{Z}_p) \times (p^{\sigma_2(s,x)} + p^{\sigma_2(s,x)+1}\mathbb{Z}_p)$$

for some definable functions $\sigma, \sigma_1, \sigma_2$ in (s, x) satisfying

$$\sigma = \sigma_1 + \sigma_2 + 1.$$

Choose an $\mathcal{L}$-definable family of functions h_s sending x in $P(\Lambda_1 \times_S \Lambda_2)_s$ to an element of $\mathbb{Q}_p$ of valuation $\sigma_2(s, x) + 1$ and with $\mathrm{ac}(h_s(x)) = 1$; such an $\mathcal{L}$-definable family exists by existence of definable Skolem functions; see [16]. Using this, we obtain an (R3)-measure-preserving bijection from $\pi_R^{-1}(x) \times \mathbb{Z}_p$ to $\pi_L^{-1}(x)$ sending (a, b) to $(\frac{a}{h_s(x)}, (b + \frac{1}{p})h_s(x))$. Gluing those bijections together for all x yields the desired (R3)-measure-preserving bijection. □

The following proposition states that a definable bijection $\Lambda_1 \to \Lambda_2$ at the value group level induces an equality between the classes $[P(\Lambda_i, \nu_i)]$ in R_S, for corresponding ν_i.

Proposition 3.15. *For $i = 1, 2$ and $n_i \in \mathbb{N}$, let $\Lambda_i \subseteq S \times \mathbb{Z}^{n_i}$ be imaginary $\mathcal{L}$-definable sets and let $\nu_i\colon \Lambda_i \to \mathbb{Z}$ be imaginary $\mathcal{L}$-definable functions. Suppose that the sets $P(\Lambda_i, \nu_i)_s$ have finite measure for each i and each s. Suppose moreover that there exists an imaginary $\mathcal{L}$-definable family of bijections $\phi_s\colon \Lambda_{1,s} \to \Lambda_{2,s}$ which is compatible with the ν_i, i.e., such that $\nu_1(s, \lambda) = \nu_2(s, \phi_s(\lambda))$ for every $(s, \lambda) \in \Lambda_1$. Then we have*

$$[P(\Lambda_1, \nu_1)] = [P(\Lambda_2, \nu_2)]$$

in R_S.

The idea of the proof is to reduce to some very simply cases, by decomposing Λ_i into finitely many pieces and by writing ϕ_s as a composition of finitely many maps. In those simple cases, explicit definable measure-preserving bijections can be obtained using the existence of definable Skolem functions. Here are the details:

Proof of Proposition 3.15. We may suppose $n_1 = n_2$, since if, say, $n_1 < n_2$, we can replace $P(\Lambda_1, \nu_1)$ by

$$P(\{-1\}^{n_2-n_1} \times \Lambda_1, \nu')_s = P(\{-1\})^{n_2-n_1} \times P(\Lambda_1, \nu_1)_s,$$

where $\nu'(s, \lambda_1, \ldots, \lambda_{n_2}) = \nu_1(s, \lambda_{n_2-n_1+1}, \ldots, \lambda_{n_2}))$. Note that $P(\{-1\})$ is just a translate of $\mathbb{Z}_p$.

Now that $n_1 = n_2 =: n$, one technique used in this proof consists in constructing an $\mathcal{L}$-definable family of C^1 bijections $\psi_s\colon P(\Lambda_1)_s \to P(\Lambda_2)_s$ such that, for every $s \in S$ and every $\lambda \in \Lambda_{1,s}$, ψ_s restricts to a bijection $P(\{\lambda\}) \to P(\{\phi_s(\lambda)\})$ and such that $v(\mathrm{Jac}\,\psi_s(x))$ is constant (and finite) on $P(\{\lambda\})$. This then induces a bijection

$$\tilde{\psi}_s\colon P(\Lambda_1)_s \times \mathbb{Q}_p \to P(\Lambda_2)_s \times \mathbb{Q}_p, \quad (x, y) \mapsto \left(\psi_s(x), \frac{y}{J_s(x)}\right),$$

where J_s is a chosen $\mathcal{L}$-definable family of C^1-functions with

$$\mathrm{ac}(J_s(x)) = 1 \quad \text{and} \quad v(J_s(x)) = v(\mathrm{Jac}\,\psi_s(x))$$

for all x in $P(\Lambda_1)_s$ (such a J exists by existence of definable Skolem functions; see [16]). An easy computation (using that the measures of $P(\{\lambda\})$ and $P(\{\phi_s(\lambda)\})$ differ by a factor $|\mathrm{Jac}\,\psi_s(x)|$, equation (3.1), and the compatibility of ϕ_s with the ν_i) shows that $\tilde{\psi}_s$ restricts to a bijection $P(\Lambda_1, \nu_1)_s \to P(\Lambda_2, \nu_2)_s$ and that this restriction satisfies (R3). Thus, the proposition follows whenever we can find ψ_s as above.

Instead of applying this technique directly in general, we first reduce to special cases where the given family $(\phi_s)_{s\in S}$ is of a simple form.

Given a partition of Λ_1 into finitely many $\mathcal{L}$-definable sets Λ_1', it suffices to prove the lemma for those Λ_1' and Λ_2' given by $\Lambda_{2,s}' = \phi_s(\Lambda_1')$. (Indeed, such a partition induces corresponding partitions of $P(\Lambda_i, \nu_i)$; then use (R1).) By piecewise linearity of Presburger functions, using such a finite partition, we may assume that ϕ_s is of the form

$$\phi_s(\lambda) = M\lambda + \mu_s$$

for some matrix $M = (m_{ij})_{ij}$ and some vector $\mu_s = (\mu_{s,i})_i$, both with coefficients in $\mathbb{Q}$, and where M does not depend on s. Since ϕ_s is a bijection, we may moreover assume that M is invertible. (Note that this might require a refinement of the finite partition.)

If we can write ϕ_s as a composition of several maps $\phi_{k,s}$ (each one forming an $\mathcal{L}$-definable family), it suffices to prove the proposition for each of the $\phi_{k,s}$. In this way, by writing the matrix M as a composition of elementary transformations, one sees that we may further assume that ϕ_s is one of the following forms:

(1) $\lambda \mapsto \lambda + \mu_s$ for some $\mu_s \in \mathbb{Z}^n$ (where $s \mapsto \mu_s$ is $\mathcal{L}$-definable),

(2) a permutation of coordinates,

(3) $(\lambda_1, \dots, \lambda_n) \mapsto (\lambda_1 + \lambda_2, \lambda_2, \dots, \lambda_n)$,

(4) $(\lambda_1, \dots, \lambda_n) \mapsto (r\lambda_1, \lambda_2, \dots, \lambda_n)$ for some $r \in \mathbb{Q}$, $r \neq 0$.

(In (2)–(4), everything is independent of s.) We now prove the proposition in each of these cases, partly by specifying a family ψ_s as required for the technique described at the beginning of the proof.

(1) By [16], there exists an $\mathcal{L}$-definable function sending $s \in S$ to an element

$$(a_1(s), \dots, a_n(s)) \in \mathbb{Q}_p^n$$

satisfying $v(a_i(s)) = \mu_{s,i}$ and $\mathrm{ac}(a_i(s)) = 1$ for each i. Then we define

$$\psi_s(x_1, \dots, x_n) = (a_1(s)x_1, \dots, a_n(s)x_n).$$

As required, this sends $P(\{\lambda\})$ to $P(\{\lambda + \mu_s\})$, and the valuation of its Jacobian is constant (namely equal to $\sum_i \mu_{s,i}$).

(2) Clear.

(3) Set $\psi_s(x_1, \dots, x_n) := (x_1 \cdot x_2, x_2, \dots, x_n)$.

(4) We may assume $r \in \mathbb{Z}$, since then we can obtain arbitrary r by composing one such map with an inverse of such a map. Also, without loss we may suppose that $n = 1$. By Hensel's Lemma one easily finds (see [1, Corollary 1]) that there exists $\ell > 0$ such that the map

$$\psi\colon x \mapsto x^r$$

defines a bijection

$$(\mathrm{ac}_\ell)^{-1}(1) \to (\mathrm{ac}_{\ell'})^{-1}(1) \cap P(r\mathbb{Z}),$$

where

$$\ell' = \ell + v(r).$$

The map ψ restricts to bijections

$$P_\ell(\{\lambda\}) \to P_{\ell'}(\{r\lambda\})$$

for each λ, and for each such restriction, the valuation of the Jacobian is constant, namely: for $x \in P_\ell(\{\lambda\})$, that valuation is equal to

$$v(rx^{r-1}) = v(r) + (r-1)\cdot\lambda.$$

By a variant of the technique from the beginning of the proof, we obtain an $\mathcal{L}$-definable bijection

$$\tilde{\psi}\colon P_\ell(\Lambda_1, \nu_1) \to P_{\ell'}(\Lambda_2, \nu_2)$$

whose Jacobian has valuation constant equal to $v(r)$. From this, one deduces that

$$[P_{\ell'}(\Lambda_2, \nu_2)] = p^{-v(r)} \cdot [P_\ell(\Lambda_1, \nu_1)].$$

Now the proposition in case (4) follows using Lemma 3.10. □

We end this section with two results from [6] about Presburger definable families that will be needed in the proof of Theorem 2.3.

To apply Proposition 3.15, we need to find definable families $\phi_s\colon \Lambda_{1,s} \to \Lambda_{2,s}$ in the value group. Those will be obtained using the following variant of Theorem 2.3 for Presburger definable sets:

Theorem 3.16 ([6, Theorem 5.2.2]). *Let $\tilde{S} \subseteq \mathbb{Z}^k$ be a Presburger set. Let $\Lambda_{\tilde{s}}$ and $\Lambda'_{\tilde{s}}$ be two Presburger families, where $\tilde{s}$ runs over $\tilde{S}$. Suppose moreover that for each $\tilde{s} \in \tilde{S}$, $\Lambda_{\tilde{s}}$ and $\Lambda'_{\tilde{s}}$ are finite sets of the same cardinality. Then there exists a Presburger family of bijections $\phi_{\tilde{s}}\colon \Lambda_{\tilde{s}} \to \Lambda'_{\tilde{s}}$, with $\tilde{s}$ running over $\tilde{S}$.*

(Recall that Presburger set is just an imaginary $\mathcal{L}$-definable subset of $\mathbb{Z}^n$, and similarly for Presburger maps.)

In the proof of Theorem 2.3, we will also need to understand how, in the above setting, the cardinality of $\Lambda_{\tilde{s}}$ can depend on $\tilde{s}$. The following proposition states that the dependence is piecewise polynomial:

Proposition 3.17 ([6, Proposition 5.2.1]). *Let $\tilde{S} \subseteq \mathbb{Z}^k$ be a Presburger set, and let $\Lambda_{\tilde{s}}$ be a Presburger family of finite sets, where $\tilde{s}$ runs over $\tilde{S}$. Then there exists a partition of $\tilde{S}$ into finitely many Presburger sets $\tilde{S}_i$ and polynomials $g_i \in \mathbb{Q}[x_1, \dots, x_k]$ such that $\#X_{\tilde{s}} = g_i(\tilde{s})$ for each $\tilde{s} \in \tilde{S}_i$.*

4. Proof of Theorem 2.3

We now have all the ingredients to prove our main result, Theorem 2.3, namely that an element Ξ of R_S is determined by the measures $\mu_s(\Xi)$, for $s \in S$. For that proof, we first introduce a subring of R_S, generated by sets that are unions of finitely many boxes, all of which have the same measure, but where the number of boxes may depend on $s \in S$. We will then first prove Theorem 2.3 for that subring (Proposition 4.3), and then show that R_S is actually not much bigger than the subring (Proposition 4.4); this will then easily imply the theorem.

Definition 4.1. Let R_S^{basic} be the subgroup of R_S generated by the classes $[P(\Lambda, \nu \circ \pi_S)]$ of sets of the form $P(\Lambda, \nu \circ \pi_S)$ for $\Lambda \subseteq S \times \mathbb{Z}^n$ and $\nu \colon S \to \mathbb{Z}$ imaginary $\mathcal{L}$-definable (for some $n \geq 0$), where $\pi_S : \Lambda \to S$ is the projection to the S-coordinates, and where moreover Λ_s is finite for every $s \in S$. We call such $P(\Lambda, \nu \circ \pi_S)$ a *basic* set, and by abuse of notation, we also denote it by $P(\Lambda, \nu)$ (thinking of ν as a function on Λ only depending on the S-variable).

Remark 4.2. By Lemma 3.13, R_S^{basic} is a subring of R_S.

Now we are ready to do the first main step of the proof of Theorem 2.3.

Proposition 4.3. *Suppose that $\Xi \in R_S^{\text{basic}}$ is an element satisfying $\mu_s(\Xi) = 0$ for every $s \in S$. Then $\Xi = 0$.*

Proof. Let $\Xi \in R_S^{\text{basic}}$ be given, satisfying $\mu_s(\Xi) = 0$ for all s in S. We need to prove $\Xi = 0$. Write Ξ as a finite sum of generators of R_S^{basic}, i.e.,

$$\Xi = \sum_j \delta_j \cdot [P(\Lambda_j, \nu_j)]$$

with the $P(\Lambda_j, \nu_j)$ basic sets and with δ_j either 1 or -1.

Given a partition of S into finitely many $\mathcal{L}$-definable sets $S_i \subseteq S$, we get natural images Ξ_i of Ξ in R_{S_i} (by Remark 3.7), and it suffices to prove that $\Xi_i = 0$ in R_{S_i} for each i. This allows us to apply cell decomposition to S, so that we may without loss assume that S equals $P_\ell(\tilde{S})$ for integer $\ell \geq 1$ and some Presburger set $\tilde{S} \subseteq \mathbb{Z}^m$ (for some m). Using that the sets $\Lambda_{j,s}$ and the values $\nu_j(s)$ live in the value group, we may moreover assume (by choosing the cell decomposition appropriately) that we have $\Lambda_{j,s_1} = \Lambda_{j,s_2}$ and $\nu_j(s_1) = \nu_j(s_2)$ whenever $v(s_1) = v(s_2)$, with v the coordinate-wise valuation map $v : S \to \tilde{S}$ as before.

We write $\Lambda_{j,\tilde{s}}$ and $\nu_j(\tilde{s})$ for $\Lambda_{j,s}$ and $\nu_j(s)$, respectively, when $s \in S$ and $\tilde{s} \in \tilde{S}$ satisfy $v(s) = \tilde{s}$. We denote the coordinates of $\tilde{s}$ by $\tilde{s}_i$ $(i = 1, \dots, m)$.

By Proposition 3.17 (and using Remark 3.7 once more to partition $\tilde{S}$), we may assume that

$$\#\Lambda_{j,\tilde{s}} = g_j(\tilde{s})$$

for some polynomials $g_j \in \mathbb{Q}[\tilde{s}_1, \dots, \tilde{s}_m]$ and that ν_j is affine linear with coefficients in $\mathbb{Q}$, i.e., of the form

$$\nu_j(\tilde{s}) = c_j + \underbrace{\sum_i b_{j,i} \cdot \tilde{s}_i}_{=: b_j \cdot \tilde{s}} \tag{4.1}$$

for some rational numbers c_j and $b_{j,i}$. Using this notation, we obtain, for $s \in S$ and $\tilde{s} = v(s)$,

$$\mu_s(\Xi) = \sum_j \delta_j \mu(P(\Lambda_j, \nu_j)) = \sum_j \delta_j \cdot g_j(\tilde{s}) \cdot p^{\nu_j(\tilde{s})},$$

so the assumption that $\mu(\Xi_s) = 0$ for all s becomes

$$\sum_j \delta_j \cdot g_j(\tilde{s}) \cdot p^{\nu_j(\tilde{s})} = 0 \quad \text{for all } \tilde{s} \in \tilde{S}. \tag{4.2}$$

By Rectilinearization [2, Theorem 2], Proposition 3.15 and a further finite partition of $\tilde{S}$, we may then assume that $\tilde{S} = \mathbb{N}^m$. After this modification, we in particular get that the g_j have integer coefficients and that the c_j and $b_{i,j}$ are integers.

We may assume that the constant terms c_j from equation (4.1) are non-negative; if not, we replace Ξ by

$$\Xi' := \sum_j \delta_j \cdot [P(\Lambda_j, \nu_j + c)]$$

for a suitable c. (Note that by Remark 3.14, we have $\Xi' = p^c \cdot \Xi$ in R_S, so $\Xi' = 0$ implies $\Xi = 0$, by torsion freeness of R_S.)

Now we can entirely get rid of the c_j: Intuitively, replacing ν_j by $\nu_j - c_j$ divides Ξ by p^{c_j}; we make up for this by replacing Λ_j by p^{c_j} disjoint copies of itself. Formally, this is the following computation (which uses Lemma 3.13 multiple times):

$$\begin{aligned}[P(\Lambda_j, \nu_j)] &= [P(\Lambda_j, \nu_j - c_j)] \cdot [P(S, c_j)] = [P(\Lambda_j, \nu_j - c_j)] \cdot p^{c_j} \\ &= [P(\Lambda_j, \nu_j - c_j)] \cdot [P(\Lambda'_j, 0)] = [P(\Lambda_j \times \Lambda'_j, \nu_j - c_j)],\end{aligned}$$

where Λ'_j a Presburger set with p^{c_j} elements.

Now that the c_j are gone, we may group summands of (4.2) together that have the same sign δ_j and the same tuple $(b_{j,1}, \dots, b_{j,m})$. Here, "grouping" summands j and j' means first making Λ_j and $\Lambda_{j'}$ disjoint (using definable bijections) and then taking their union. In this way, (4.2) becomes, after some relabeling,

$$\sum_j (g_{j+}(\tilde{s}) - g_{j-}(\tilde{s})) \cdot p^{b_j \cdot \tilde{s}} = 0 \quad \text{for all } \tilde{s} \in \tilde{S} = \mathbb{N}^m, \tag{4.3}$$

where $b_j \neq b_{j'}$ for $j \neq j'$. Using a suitable induction according to growth rates of the sum (as a function of the $\tilde{s}_i$), (4.3) implies $g_{j+} = g_{j-}$ for all j. In particular, for every $\tilde{s} \in \tilde{S}$, the corresponding sets $\Lambda_{j+,\tilde{s}}$ and $\Lambda_{j-,\tilde{s}}$ have the same cardinality. By Theorem 3.16, this implies that there exists an $\mathcal{L}$-definable family of bijections $\phi_s\colon \Lambda_{j+,\tilde{s}} \to \Lambda_{j-,\tilde{s}}$. Then Proposition 3.15 yields $[P(\Lambda_{j+}, \nu_j)] = [P(\Lambda_{j-}, \nu_j)]$, which, by summing over j, implies $\Xi = 0$. □

Proposition 4.4. *For every $\Xi \in R_S$, there is an integer $\ell > 0$ such that $\ell \cdot \Xi \in R_S^{\mathrm{basic}}$.*

Proof. Since every element of R_S can be written as a difference of two generators (by Remark 3.6), it suffices to deal with the case $\Xi = [X]$, for some $\mathcal{L}$-definable $X \subseteq S \times \mathbb{Q}_p^n$ (satisfying that X_s has finite p-adic measure for each $s \in S$).

By cell decomposition (and since R_S^{basic} is closed under finite sums), we may assume that X is a cell over S in the sense of, e.g., [5, Definition 3.4] (which ultimately originates from [9, proof of Theorem 7.4]). We may also assume that $\dim X_s = n$ for each $s \in S$, by (R2) and

Remark 3.7 to partition S if necessary. Moreover, we may get rid of the centers of the cell by (R3)-measure-preserving translations. In this way, we reduce to the case where X is of the form

$$X = \{(s,x) \in S \times \mathbb{Q}_p^n \mid v(x) \in \Lambda_s, \mathrm{ac}_\ell(x_1) = \xi_1, \dots, \mathrm{ac}_\ell(x_n) = \xi_n\}$$

for an imaginary $\mathcal{L}$-definable set $\Lambda \subseteq S \times \mathbb{Z}^n$, an integer $\ell > 0$ and some $\xi_i \in \mathbb{Z}_p/p^\ell\mathbb{Z}_p$ (where $x = (x_1, \dots, x_n)$ and $v(x) = (v(x_1), \dots, v(x_n))$). Moreover, we can replace all ξ_i by 1 (by multiplying each coordinate by a suitable element of $\mathbb{Z}_p^\times$), so that $X = P_\ell(\Lambda)$. Finally, Lemma 3.10 allows us to reduce to the case where $\ell = 1$ (so that $X = P(\Lambda)$).

Next, we replace X by $P(\Lambda, \nu)$, where $\nu\colon \Lambda \to \mathbb{Z}$ is chosen in such a way that

$$P(\Lambda, \nu) = P(\Lambda) \times (p^{-1} + \mathbb{Z}_p)$$

(which implies $[P(\Lambda)] = [P(\Lambda, \nu)]$).

Given a partition of Λ into finitely many definable pieces, it suffices to prove the proposition for each piece. Moreover, given a definable bijection $\phi\colon \Lambda' \to \Lambda$, compatible with the projection to S, Proposition 3.15 allows us to replace $P(\Lambda, \nu)$ by $P(\Lambda', \nu \circ \phi)$. In the following, we will apply those two techniques several times to simplify Λ.

By applying the Parametric Rectilinearization [2, Theorem 3] to Λ, we reduce to the case where Λ is of the form $\Lambda_0 \times \mathbb{N}^m$ for some non-empty $\Lambda_0 \subseteq S \times \mathbb{Z}^{n-m}$ having finite fibers $\Lambda_{0,s}$, and where moreover the function $\nu\colon \Lambda \to \mathbb{Z}$ is of the form

$$\nu(s, \lambda_1, \dots, \lambda_n) = \frac{1}{r}(c(s) + \sum_i b_i \lambda_i) \tag{4.4}$$

for some integers $r \geq 1$ and b_i and some definable $c\colon S \to \mathbb{Z}$. Using that ν takes integer values on all of Λ, we deduce that b_i is a multiple of r for each $i > n - m$. Then, Lemma 3.13 allows us to write $[P(\Lambda, \nu)]$ as a product

$$[P(\Lambda_0, \nu_0)] \cdot \prod_{i=n-m+1}^{n} [P(S \times \mathbb{N}, \nu_i)]$$

for $\nu_i(s, \lambda_i) = \frac{b_i}{r}\lambda_i$ and $\nu_0(s, \lambda_1, \dots, \lambda_{n-m}) = \frac{1}{r}(c(s) + \sum_{i=1}^{n-m} b_i \lambda_i)$. Thus it remains to treat the following two cases:

Case 1. One has $X = P(S \times \mathbb{N}, \nu)$ with $\nu(s, \lambda) = b\lambda$ for some $b \in \mathbb{Z}$.

Since nothing depends on s, we omit s and S in the proof of this case.

The p-adic measure of X is equal to $\sum_{\lambda \in \mathbb{N}} p^{b\lambda}$, so this measure being finite implies $b < 0$. Applying Proposition 3.15 to the diagonal embedding

$$\Delta\colon \mathbb{N} \to \mathbb{Z}^n, \quad \lambda \mapsto (\lambda, \dots, \lambda)$$

for $n := -b$ yields that $[X]$ equals the class of $P(\Delta(\mathbb{N}), (\lambda, \dots, \lambda) \mapsto \nu(\lambda)))$. By definition (and using $\nu(\lambda) = -n\lambda$), this set is equal to

$$\{(x, y) \in P(\Delta(\mathbb{N})) \times \mathbb{Q}_p \mid \mathrm{ac}(y) = 1,\ v(y) = -n - 1\},$$

which is the Cartesian product of $P(\Delta(\mathbb{N}))$ with a translate of $p^{-n}\mathbb{Z}_p$. Now we conclude using Lemma 3.9:

$$(p^n - 1)[X] = (p^n - 1)[P(\Delta(\mathbb{N}))] \cdot [p^{-n}\mathbb{Z}_p] = 1 \cdot p^n \in R^{\mathrm{basic}}.$$

Case 2. One has $X = P(\Lambda, \nu)$, where $\Lambda \subseteq S \times \mathbb{Z}^n$ has finite fibers over S and

$$\nu(s, \lambda_1, \dots, \lambda_n) = \frac{1}{r}\Big(c(s) + \sum_i b_i \lambda_i\Big) \tag{4.5}$$

with r, c, b_i as in (4.4).

If all b_i are 0, then X is a basic set and we are done, so suppose without loss that $b_n \neq 0$. We do an induction on n, i.e., we will reduce this to a case where $X = P(\Lambda', \nu')$ for some $\Lambda' \subseteq S \times \mathbb{Z}^{n-1}$ with finite fibers over S. Note that if $n = 0$, then clearly X is basic.

Without loss of generality, $b_n < 0$. (Otherwise, multiply the λ_n-coordinate by -1.) By cell-decomposing Λ with respect to the last variable, we may assume that it is of the form

$$\Lambda = \{(\hat{\lambda}, \lambda_n) \in \hat{\Lambda} \times \mathbb{Z} \mid \alpha_1(\hat{\lambda}) \leq \lambda_n < \alpha_2(\hat{\lambda}),\ \lambda_n \equiv k \bmod \ell\}$$

for some integers k, ℓ with $0 \leq k < \ell$, some definable $\hat{\Lambda} \subseteq S \times \mathbb{Z}^{n-1}$ and some definable $\alpha_1, \alpha_2 \colon \hat{\Lambda} \to \mathbb{Z}$ satisfying $\alpha_1(\hat{\lambda}) < \alpha_2(\hat{\lambda})$ for all $\hat{\lambda} \in \hat{\Lambda}$. By a further finite partition and a linear transformation onto Λ (namely sending λ_n to $\ell \cdot \lambda_n + k$), we may get rid of the congruence condition, so that the (new) set Λ is equal to the set-theoretic difference $\Lambda_1 \setminus \Lambda_2$, where

$$\Lambda_i = \{(\hat{\lambda}, \lambda_n) \in \hat{\Lambda} \times \mathbb{Z} \mid \alpha_i(\hat{\lambda}) \leq \lambda_n\}$$

for $i = 1, 2$. A similar partition and transformation allows us to moreover assume that in equation (4.5), b_n is divisible by r, so that (4.5) can be used to extend the definition of ν to the larger domain Λ_1. Using this extension of ν, we set $X_i := P(\Lambda_i, \nu)$. Since $b_n < 0$, the sets $X_{i,s}$ have finite measure and we have the equation $[X] = [X_1] - [X_2]$ in R_S. In this way, we reduced the problem of proving that a multiple of $[X]$ lies in R_S^{basic} to proving it for both $[X_i]$. Using a linear transformation, we reduce to the case $\alpha_i = 0$, so that $\Lambda_i = \Lambda_i' \times \mathbb{N}$ (for some $\Lambda_i' \subseteq S \times \mathbb{Z}^{n-1}$ with finite fibers over S). As in the above discussion just before Case 1, we now can write $[X_i]$ as a product $[P(\Lambda_i', \nu_i')] \cdot [P(\mathbb{N}, \nu_i'')]$. The first factor is treated by the induction on n in Case 2, and the second one has already been treated before, in Case 1. This finishes the proof of the proposition. □

Proof of Theorem 2.3. Let $\Xi \in R_S$ be given such that $\mu_s(\Xi) = 0$ for every $s \in S$. We need to prove $\Xi = 0$.

By Proposition 4.4, we find an integer $\ell \geq 1$ such that $\ell \cdot \Xi \in R_S^{\text{basic}}$. Then we also have $\mu_s(\ell \cdot \Xi) = 0$ for every $s \in S$, so by Proposition 4.3, we obtain $\ell \cdot \Xi = 0$. Now $\Xi = 0$ follows from R_S being torsion free (Lemma 3.11). □

Remark 4.5. The only ingredients used in the entire paper are cell decomposition (which implies dimension theory and related results), the existence of definable Skolem functions and the fact that the (imaginary) $\mathcal{L}$-definable subsets of $\mathbb{Z}^n$ are exactly the Presburger sets. Therefore, all our results (notably Theorem 2.3) also hold in various generalized situations where we have those ingredients. In particular:

(1) Instead of $\mathcal{L}$ as specified at the beginning of Section 2 (as parameter free language of valued fields), one can take any language which expands $\mathcal{L}$ with an analytic structure on $\mathbb{Q}_p$, as in [7]. (The proof ingredients hold in this generality by [7].) In particular,

the map from Theorem 2.3 is still injective if we define the ring R_S using subanalytic sets on $\mathbb{Q}_p$, in the sense of [11]. Indeed, [7] provides cell decomposition results (with preparation for definable functions) for these cases similar to [10] for the semi-algebraic case and to [3] for the subanalytic case.

(2) Instead of $\mathbb{Q}_p$, any finite field extension K of $\mathbb{Q}_p$ can be used, provided that one expands the language $\mathcal{L}$ with a constant symbol for a uniformizing element ϖ and enough constant symbols to obtain definable Skolem functions. (And again, one can expand $\mathcal{L}$ by an analytic structure.) To make the proofs work in K, most occurrences of p need to be replaced either by ϖ or by the cardinality of the residue field. The least straightforward changes might be those to the proof of Lemma 3.11: There, we obtain $[\varpi\mathcal{O}_K]$ as a multiplicative inverse of q (which then yields that p is invertible since $p \mid q$), and to get that $\ell \neq p$ is invertible, we use $\ell \mid (p^n - 1) \mid (q^n - 1)$, and invertibility of $q^n - 1$ follows from Lemma 3.9.

(3) Even more generally, one can use any language $\mathcal{L}$ such that the $\mathcal{L}$-theory of K is hensel minimal (more precisely, ω-h$^{\mathrm{eqc}}$-minimal as defined in [4, Section 6.1]), with pure Presburger structure on the value group and with definable Skolem functions on K. (For cell decomposition with preparation, see [4, Theorem 5.2.4 and Addendum 1])

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